# Closed-Form Expressions for the Current Distributions on Open Microstrip Lines

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*Abstract* —**An open microstrip line with isotropic dielectric substrate is numerically analyzed by the spectral-domain method to clarify current distributions on a strip conductor. The current distributions obtained are illustrated in figures for typical cases. A full view of current distributions on a strip conductor in any microstrip line is provided by showing those closed-form expressions.**

## I. Introduction

The spectral-domain analysis proposed by Itoh and Mittra [1] is very powerful in calculating numerically the dispersion characteristics of microstrip lines. It is well known that the effective relative permittivities can be obtained with a high degree of accuracy by the spectral-domain analysis (see [2]–[6] and references therein). Even a small number of basis functions for the expression of current distributions give accurate results for effective relative permittivities, $\epsilon_{\text{eff}}(f)$. However, to accurately obtain the current distributions requires a large number of basis functions and thus more CPU time. Therefore, the literature on the determination of these distributions is sparse.

Recently, Shih *et al.* [3] revealed the frequency dependence of the current distributions on the microstrip lines. This was the first time that those characteristics were reasonably obtained for wide ranges of frequency. Subsequently, the frequency-dependent characteristics of the current distributions were published by Faché and De Zutter [4], Kobayashi and Iijima [5], and Uchida *et al.* [6].

However, there is no published work to clarify systematically the characteristics of current distributions for various cases of substrate relative permittivity, $\epsilon$, shape ratio, $w/h$, and normalized frequency $h/\lambda_0$.

The present article seeks to clarify these characteristics and proposes closed-form expressions. The results obtained are meaningful from both practical and educational points of view since the current distributions are fundamental quantities as sources for electromagnetic fields in microstrip lines.

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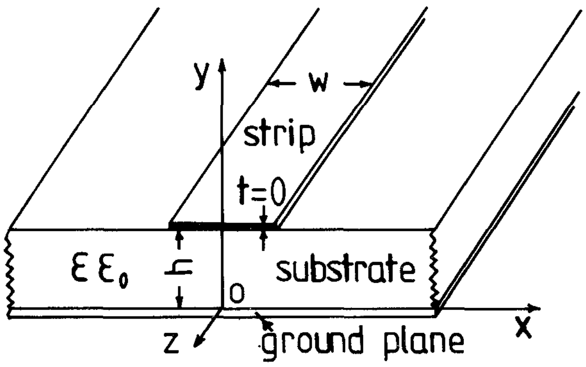


Fig. 1. Open microstrip line with isotropic dielectric substrate.

## II. Method of Calculation

The open microstirp line under consideration is shown in Fig. 1. It is assumed to be uniform and infinite in both the $x$ and the $z$ direction. The infinitesimally thin strip and the ground plane are taken as perfect conductors. It is also assumed that the substrate material is lossless and that its relative permittivity and permeability are $\epsilon$ and $\mu$ $(=1)$, respectively.

The present article uses Chebyshev polynomials of the first and second kinds, $T_n(x)$ and $U_n(x)$, as basis functions for the expression of current components and takes the following functions for $I_{xn}(x)$ and $I_{zn}(x)$:

$$I_{xn}(x) = U_{2n}(2x/w) \tag{1a}$$

$$I_{zn}(x) = T_{2(n-1)}(2x/w)/\sqrt{-(2x/w)^2} \tag{1b}$$

on the strip $(|x| \leqslant w/2)$ and zero for $|x| > w/2$. Let Fourier-transformed functions of these be expressed by $\tilde{I}_{xn}(\alpha)$ and $\tilde{I}_{zn}(\alpha)$, respectively (see [5]). Using these $\tilde{I}_{xn}(\alpha)$ and $\tilde{I}_{zn}(\alpha)$, expanding the current components $\tilde{I}_x(\alpha)$ and $\tilde{I}_z(\alpha)$ as

$$\tilde{I}_x(\alpha) = \sum_{n=1}^{M} c_n \tilde{I}_{xn}(\alpha) \tag{2a}$$

$$\tilde{I}_z(\alpha) = \sum_{n=1}^{N} d_n \tilde{I}_{zn}(\alpha) \tag{2b}$$

and using the method given in [1], following the corrections shown in [2], we can obtain the following simultaneous equations:

$$\sum_{n=1}^{M} K_{mn}(1,1)c_n + \sum_{n=1}^{N} K_{mn}(1,2)d_n = 0,$$
$$m = 1,2,\cdots,N \quad (3a)$$

$$\sum_{n=1}^{M} K_{mn}(2,1)c_n + \sum_{n=1}^{N} K_{mn}(2,2)d_n = 0,$$
$$m = 1,2,\cdots,M \quad (3b)$$

where $K_{mn}(i,j)$ are Fourier integrations and are given in [1].

The phase constant $\beta$ can be solved by setting the determinant of the simultaneous equations (3) equal to zero and seeking the root of the resulting equation.

The numerical calculations of the integrations of $K_{mn}(i,j)$ are actually performed from zero to an upper limit $\alpha_u$ with respect to an integral variable $\alpha$ owing to limits on computation time. We take $\alpha_u = 7\times10^3/w$ in the present article except for the particular cases of smaller EFX shown below in (11), for example $\alpha_u = 3\times10^4/w$ for the case of $\epsilon = 8$, $w/h = 0.1$, $h/\lambda_0 = 0.05$, and $M = N = 3$. The present article took $M = N = 3$ for $h/\lambda_0 \leqslant 0.2$, $M = N = 4$ for $0.2 < h/\lambda_0 \leqslant 0.4$, and $M = N = 5$ to 9 for $h/\lambda_0 > 0.4$ to accurately obtain the current distributions.

First of all, the present article seeks the root $\epsilon_{\text{eff}}(f)$ of the resulting equation which is obtained by setting the determinant of the simultaneous equations (3) for the case of $M = N = 3$ equal to zero.

Secondly, substituting the value obtained for $\epsilon_{\text{eff}}(f)$ into the simultaneous equations (3) for the case of larger $M$ and $N$, we get the resulting simultaneous equations (3) for unknown coefficients $c_n$ $(n = 1,2,\cdots,M)$ and $d_n$ $(n = 1,2,\cdots,N)$. Solving these equations, we can determine unknown coefficients $c_n/c_1$ $(n = 2,3,\cdots,M)$ and $d_n/c_1$ $(n = 1,2,\cdots,N)$ normalized to $c_1$ for obtaining the current distributions. This procedure reduces CPU time in obtaining the current distributions.

## III. Numerical Results

Shih *et al.* [3] showed the shape-ratio-dependent characteristics of the normalized longitudinal current distributions for six cases of $w/h$ $(= 0.1, 0.4, 1.043, 2, 4, 10)$ when $\epsilon = 11.7$, and $h/\lambda_0 = 0.01$. Fig. 2 shows the shape-ratio-dependent characteristics of the normalized longitudinal and transverse current distributions ($\epsilon = 8$, $h/\lambda_0 = 0.4$, $w/h = 0.1, 0.4, 1, 2$) obtained in the present article by solid lines. The dotted lines show the results by the approximate formulas (4) and (8). The results shown by Shih *et al.* [3] were for the cases of lower frequency, $h/\lambda_0 = 0.01$; the results shown in Fig. 2 are for the cases of higher frequency, $h/\lambda_0 = 0.4$.

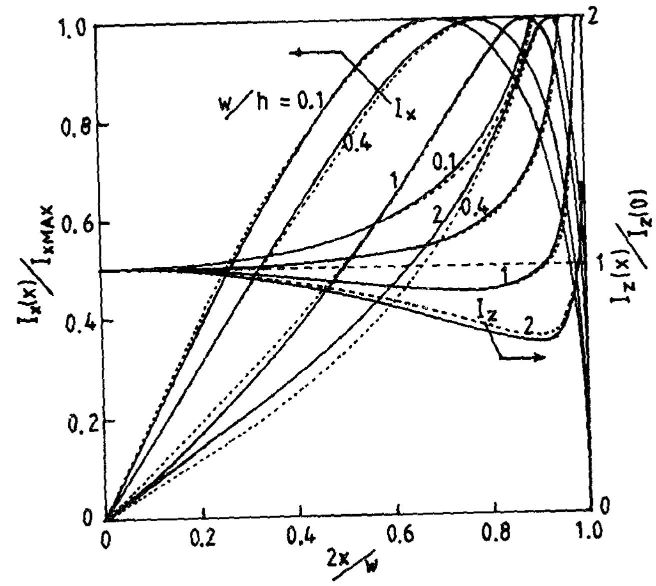


Fig. 2. Shape-ratio-dependent characteristics of normalized longitudinal and transverse current distributions ($\epsilon = 8$, $h/\lambda_0 = 0.4$). $h/\lambda_0 = h(\text{mm})f(\text{GHz})/299.7925$. —— theoretical results; ···· approximate formulas (4) and (8).

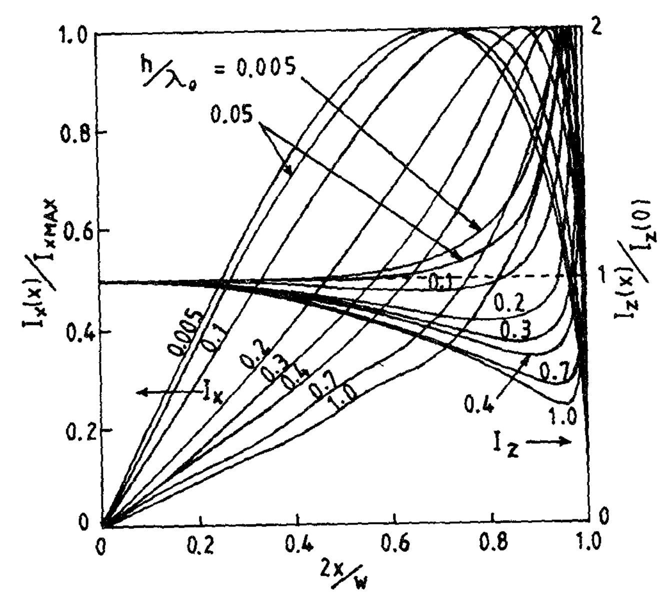


Fig. 3. Frequency-dependent characteristics of normalized longitudinal and transverse current distributions ($\epsilon = 8$, $w/h = 2$).

Fig. 3 shows the frequency-dependent characteristics of the normalized longitudinal and transverse current distributions for the case where $\epsilon = 8$ and $w/h = 2$. Comparing those results with the results for $w/h = 1$ shown in [5], it is seen that the form of the curve for $w/h = 2$ changes more rapidly as a function of frequency than that for $w/h = 1$.

Fig. 4 shows the relative-permittivity-dependent characteristics of the normalized longitudinal and transverse current distributions for the case where $w/h = 1$ and $h/\lambda_0 = 0.2$. Shih *et al.* [3] also showed the relative-permittivity-dependent characteristics for lower frequency, $h/\lambda_0 = 0.00424$, so that the curves were barely distin-

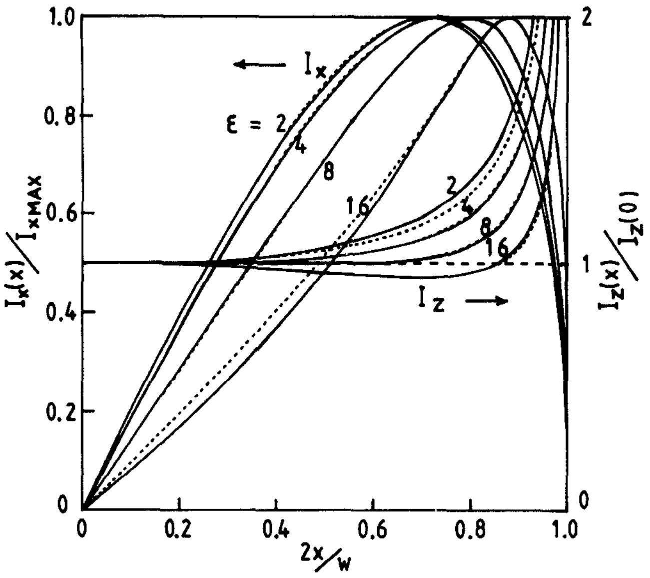


Fig. 4. Relative-permittivity-dependent characteristics of normalized longitudinal and transverse current distributions ($w/h=1$, $h/\lambda_0=0.2$). ——— theoretical results; ····· approximate formulas (4) and (8).

guishable for different relative permittivities. However, it is seen in Fig. 4 that the characteristics of the current distributions depend to a marked extent on the relative permittivity of the substrate at higher frequency.

The present article analyzed numerically and systematically the various cases of $\epsilon = 2, 4, 8, 16$, $0.1 \leqslant w/h \leqslant 10$, and $0 < h/\lambda_0 \leqslant 1$ and provides a complete view of current distributions on open microstrip line with isotropic dielectric substrate, as shown in following sections. This can be performed by paying particular attention to the upper limit $\alpha_u$ with respect to the integral variable $\alpha$ in Fourier integrations and to the numbers $M$ and $N$ of the basis functions for current distributions, as mentioned above.

## IV. Closed-Form Expression for Longitudinal Current Distribution

One of the present authors previously proposed a closed-form expression for the frequency-dependent characteristics of the longitudinal current distribution for the case where $\epsilon = 8$ and $w/h = 1$ [7]. Good agreement was found between the closed-form expression and the theoretical results [7]. The present authors could modify the expression for the cases of various $\epsilon$, $w/h$, and $h/\lambda_0$ by taking a best fit to theoretical results obtained in the present paper.

We propose the closed-form expression for the normalized longitudinal current distribution as follows:

$$\frac{I_z(x)}{I_z(0)} = \frac{1 - A\left(\frac{2x}{w}\right)^2}{\sqrt{1-\left(\frac{2x}{w}\right)^2}} \qquad (4)$$

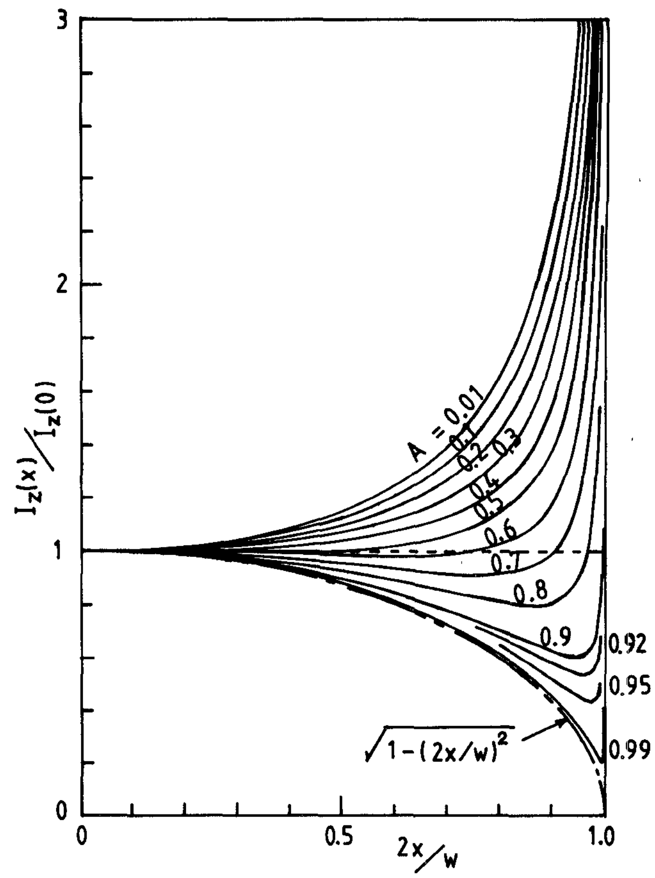


Fig. 5. $A$-dependent characteristics of normalized longitudinal current distributions by the present formula, (4).

where

$$A = 1 - \frac{1}{\left\{1+\left(\frac{w/h}{2.4}\right)^{1.66}\right\}\left\{1+\left(\frac{\mathrm{EFZ}}{0.24}\right)^{b}\right\}} \qquad (5)$$

$$b = 1.09 - 0.225\log_{10}(h/\lambda_0) \qquad (\geqslant 1) \qquad (6)$$

$$\mathrm{EFZ} = (\epsilon/8)^{0.66}(w/h)^{0.85}(h/\lambda_0). \qquad (7)$$

The major magnitude of parameter $A$ is determined by the shape ratio $w/h$. The relative permittivity dependence of parameter $A$ becomes extremly small at lower frequency. This explains why the curves could be distinguished from one another only with difficulty for different relative permittivities in the study by Shih *et al.* [3].

It is meaningful from practical and educational points of view to illustrate and explain the parameter-$A$-dependent characteristics of the normalized longitudinal current distributions for various values of $A$ in one figure. The $A$-dependent characteristics are computed by the present formula (4) and are illustrated in Fig. 5.

The distribution curve begins to have a part below the horizontal line of $I_z(x)/I_z(0)=1$ (shown by the dashed line in Fig. 5) for parameter $A$ higher than 0.5 when $w/h<2.4$. For $w/h \geqslant 2.4$, the distribution curve has a part below the horizontal line for $\mathrm{EFZ}>0$ and that is for $h/\lambda_0 > 0$. The distribution curve approaches the curve of

$\sqrt{1-(2x/w)^2}$ shown by the dot–dash line, except the edge point, when parameter $A \Rightarrow \infty$ and has the singularity $1/\sqrt{1-(2x/w)^2}$, at any parameter $A$, at the edge of the strip.

The usefulness of the present closed-form expression is confirmed by good agreement between the theoretical results (solid lines) and the results obtained with the present formula (4) (dotted lines) in Figs. 2 and 4.

## V. Closed-Form Expression for Transverse Current Distribution

The present authors could not approximate the normalized transverse current distributions for various cases of $\epsilon$, $w/h$, and $h/\lambda_0$ in simple form. This is mainly because large numbers of basis functions are needed to accurately express the transverse current distributions. However, we developed the closed-form expression for those distributions, although not in simple form.

We propose a closed-form expression for the normalized transverse current distribution as follows:

$$\frac{I_x(x)}{I_{x\max}} = c_1 U_2(x) + c_2 U_4(x) + c_3 U_6(x) + c_4 U_8(x) + c_5 U_{10}(x) \quad (8)$$

where

$$\frac{c_2}{c_1} = 1 - \frac{1}{1+\left(\dfrac{\mathrm{EFX}-0.04}{0.62}\right)^g} \quad (9a)$$

$$\frac{c_3}{c_1} = 1 - \frac{1}{1+\left(\dfrac{\mathrm{EFX}-0.147}{1.65}\right)^g} \quad (9b)$$

$$\frac{c_4}{c_1} = 1 - \frac{1}{1+\left(\dfrac{\mathrm{EFX}-0.304}{3.57}\right)^g} \quad (9c)$$

$$\frac{c_5}{c_1} = 1 - \frac{1}{1+\left(\dfrac{\mathrm{EFX}-0.44}{8.11}\right)^g} \quad (9d)$$

$$g = 1.09 - 0.225\log_{10}\mathrm{EFX} \qquad (\geqslant 1) \quad (10)$$

$$\mathrm{EFX} = \begin{cases} (\epsilon/8)^{1.1}(w/h)^{0.9}(h/\lambda_0) & (\epsilon \leqslant 8) \quad (11a) \\ (\epsilon/8)^{0.9}(w/h)^{0.9}(h/\lambda_0) & (\epsilon > 8). \quad (11b) \end{cases}$$

Here $c_2/c_1 = 0$ for EFX $< 0.04$, $c_3/c_1 = 0$ for EFX $< 0.147$, $c_4/c_1 = 0$ for EFX $< 0.304$, and $c_5/c_1 = 0$ for EFX $< 0.44$. Unknown $c_1$ should be determined as a value letting $I_x(x)/I_{x\max} = 1$ be satisfied at a maximum point of the distribution curve obtained by (8).

Fig. 6 shows the EFX-dependent characteristics of the normalized transverse current distributions computed by the present formula, (8). Good agreement is seen between the theoretical results (solid lines) and the results obtained by the present formula (8) (dotted lines) in Fig. 2 and Fig. 4.

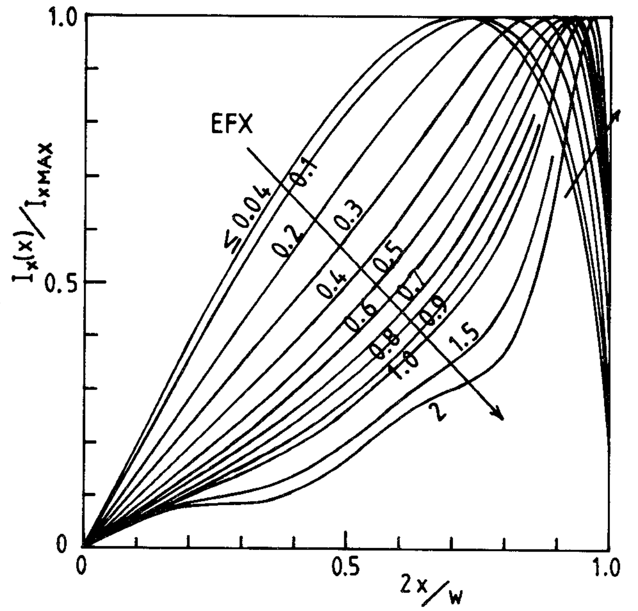


Fig. 6. EFX-dependent characteristics of normalized transverse current distributions by the present formula, (8).

## VI. Conclusions

Current distributions have been computed for open microstrip lines with isotropic dielectric by spectral-domain analysis. A complete view of those distributions has been provided by proposing closed-form expressions. These expressions for the current distribution have been compared with the theoretical results, and good agreement has been seen.

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